Gröbner bases of toric ideals and their application

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Gröbner bases and toric ideals

Gröbner bases

- A “very good” set of polynomials
- keyword: division of a polynomial (by several polynomials in $n$ variables.)
- invented by B. Buchberger in 1965. (“standard bases” H. Hironaka in 1964.)
- Elimination Theorem for systems of polynomial equations
- implemented in a lot of mathematical software Mathematica, Maple, Macauley2, Singular, CoCoA, Risa/Asir, ....
Gröbner bases and toric ideals

Toric ideals

- Prime ideals generated by binomials
- Gröbner bases of toric ideals have a lot of application
  - commutative algebra, algebraic geometry
  - triangulations of convex polytopes
  - integer programming
  - contingency tables (statistics)
  - ...
System of linear equations

Example

\[
\begin{align*}
  f_1 &= x_1 + x_3 + 3x_4 &= 0 \\
  f_2 &= x_2 - x_3 - 2x_4 &= 0 \\
  f_3 &= 2x_1 + 3x_2 - x_3 &= 0
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 0 & 1 & 3 \\
  0 & 1 & -1 & 2 \\
  2 & 3 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 1 & 3 \\
  0 & 1 & -1 & 2 \\
  0 & 3 & -3 & -6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 1 & 3 \\
  0 & 1 & -1 & 2 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

\[f_3 = 2f_1 + 3f_2.\]
Division?

\[
\frac{x^2 + 1}{x - 1} = \frac{x^2}{x - 1} - \frac{x}{x - 1}
\]

For example, which monomial in

\[f = x_1^2 + 2x_1x_2x_3 - 3x_1 + x_3^5 + 5\]

should be the largest?
Monomial order

**Definition**

\( \mathcal{M}_n \): set of all monomials in the variables \( x_1, \ldots, x_n \)

A total order \( < \) on \( \mathcal{M}_n \) is called a **monomial order** if \( < \) satisfies the following:

1. \( u \in \mathcal{M}_n, \ u \neq 1 \implies 1 < u. \)
2. \( u, v, w \in \mathcal{M}_n, \ u < v \implies uw < vw. \)

\[
\begin{align*}
&x - 1 \\
\frac{x^2}{x} + 1 \\
\frac{x^2}{x} - x \\
\frac{x}{x} \\
\frac{x}{x - 1} \\
\frac{-x^2}{x} - x^3 \\
\frac{x^2}{x} - x^3 \\
\frac{x^3}{x^3} - x^4 \\
\frac{x^4}{x^4} \\
\vdots
\end{align*}
\]
Lexicographic order

Example (Lexicographic order \((x_1 > \cdots > x_n)\))

\[ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \iff\]

\[ a_1 > b_1 \]

or

\[ a_1 = b_1 \text{ and } a_2 > b_2 \]

or

\[ a_1 = b_1, \ a_2 = b_2, \text{ and } a_3 > b_3 \]

or

\[ \vdots \]

For example,

\[ x_1 >_{\text{lex}} x_2^{100} x_3 \]

\[ x_1^2 x_2^2 x_5 >_{\text{lex}} x_1^2 x_2 x_3 x_4 \]
(Degree) Reverse lexicographic order

Example (Reverse lexicographic order \((x_1 > \cdots > x_n)\))

\[ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\text{revlex}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \iff \]

\[ \sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i \]

or

\[ \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \text{ and } a_n < b_n \]

or

\[ \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i, \quad a_n = b_n \text{ and } a_{n-1} < b_{n-1} \]

\[ \vdots \]

For example,

\[ x_1 <_{\text{revlex}} x_2^{100} x_3 \]

\[ x_1^2 x_2^2 x_5 <_{\text{revlex}} x_1^2 x_2 x_3 x_4 \]
Weight order

Example (Weight order $>_w$)

$w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_{\geq 0}$

$<: \text{ a monomial order (for "tie break")}$

$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_w x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ \iff

$$\sum_{i=1}^{n} a_i w_i > \sum_{i=1}^{n} b_i w_i$$

or

$$\sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} b_i w_i \text{ and } x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} > x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$
Remark

If \( n = 1 \), then a monomial order is unique.
In fact, if \( < \) is a monomial order on monomials in \( x_1 \), then

\[
x_1 > 1 \implies x_1^2 > x_1 \implies x_1^3 > x_1^2 \implies x_1^4 > x_1^3 \implies \cdots
\]

Hence, we have

\[
1 < x_1 < x_1^2 < x_1^3 < \cdots
\]

If \( n \geq 2 \), then there exist infinitely many monomial orders.
Initial monomial

\( K[x_1, \ldots, x_n] \): polynomial ring over a field \( K \) (e.g., \( K = \mathbb{C} \))
- The set of all polynomials in variables \( x_1, \ldots, x_n \) with coefficients in \( K \).

Fix a monomial order \(<\) on \( K[x_1, \ldots, x_n] \).

\( 0 \neq f \in K[x_1, \ldots, x_n] \)

\( \text{in}_{<}(f) \): the largest monomial among the monomials in \( f \)

**Example**

\( >_{\text{lex}} \): lexicographic order (\( x_1 > \cdots > x_5 \))

\( >_{\text{revlex}} \): reverse lexicographic order (\( x_1 > \cdots > x_5 \))

\[ f = x_1^2 + 2x_1x_2x_3 - 3x_1 + x_3^5 + 5 \]

Then, \( \text{in}_{>_{\text{lex}}} (f) = x_1^2 \) and \( \text{in}_{>_{\text{revlex}}} (f) = x_3^5 \)
Theorem (Division algorithm)

\( <: \text{monomial order} \)

\( 0 \neq f, g_1, g_2, \ldots, g_s \in K[x_1, \ldots, x_n] \)

Then, there exist \( f_1, \ldots, f_s, r \in K[x_1, \ldots, x_n] \) such that

- \( f = f_1 g_1 + f_2 g_2 + \cdots + f_s g_s + r. \)
- If \( r \neq 0 \), then any monomial in \( r \) is divided by none of \( \text{in}_<(g_1), \ldots, \text{in}_<(g_s). \)
- If \( f_i \neq 0 \), then \( \text{in}_<(f) \geq \text{in}_<(f_i g_i). \)

\( r \) is called a remainder of \( f \) w.r.t. \( \{g_1, \ldots, g_s\}. \)
Example

$n = 2$, lexicographic order \((x > y)\)

\[
\begin{array}{c c c c c}
\hline
x & + & y & & 1 \\
\hline
\hline
xy - 1 & \quad & x^2y & + & xy^2 & + & y^2 \\
\hline
y^2 - 1 & \quad & x^2y & - & x & & \\
\hline
\quad & \quad & xy^2 & + & x & + & y^2 \\
\quad & \quad & xy^2 & - & y & & \\
\hline
\quad & \quad & x & + & y^2 & + & y \\
\quad & \quad & y^2 & - & 1 & & \\
\hline
\quad & \quad & x & + & y & + & 1 \\
\end{array}
\]

\[x^2y + xy^2 + y^2 = (x + y)(xy - 1) + 1 \cdot (y^2 - 1) + x + y + 1\]
Example

\( n = 2, \) lexicographic order \((x > y)\)

\[
\begin{array}{ccc}
x & & +1 \\
x y - 1 & & \\
y^2 - 1 & &\\n\end{array}
\]

\[
\begin{array}{ccc}
& x^2 y & + xy^2 & + y^2 \\
& x^2 y & - x & \\
& & xy^2 & + x & + y^2 \\
& & xy^2 & - x & \\
& & 2x & + y^2 & \\
& & & y^2 & - 1 \\
\end{array}
\]

\[ x^2 y + xy^2 + y^2 = x \cdot (xy - 1) + (x + 1) \cdot (y^2 - 1) + 2x + 1 \]
Ideals of polynomial rings

**Definition**

Let \( f_1, \ldots, f_s \in K[x_1, \ldots, x_n] \). Then, we define

\[
\langle f_1, \ldots, f_s \rangle := \{ h_1 f_1 + \cdots + h_s f_s \mid h_i \in K[x_1, \ldots, x_n] \}
\]

ideal generated by \( f_1, \ldots, f_s \in K[x_1, \ldots, x_n] \).

**Proposition**

If \( \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle \), then

\[
\begin{align*}
\begin{cases}
  f_1 = 0 \\
  \vdots \\
  f_s = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
  g_1 = 0 \\
  \vdots \\
  g_t = 0
\end{cases}
\end{align*}
\]

have the same solutions.
Fix a monomial order $\prec$.

$f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$

$I = \langle f_1, \ldots, f_s \rangle \subset K[x_1, \ldots, x_n]$: ideal

**Definition**

\[ \text{in}_\prec(I) := \langle \text{in}_\prec(f) \mid 0 \neq f \in I \rangle \] the initial ideal of $I$

**Definition**

\[ \{g_1, \ldots, g_t\} \subset I \text{ is a Gröbner basis of } I \text{ w.r.t. } \prec \]

\[ \text{def} \quad \text{in}_\prec(I) = \langle \text{in}_\prec(g_1), \ldots, \text{in}_\prec(g_t) \rangle \]

\[ \iff \quad \text{For any nonzero element } f \in I, \]

\[ \text{in}_\prec(f) \text{ is divided by } \text{in}_\prec(g_i) \text{ for some } i. \]
Example

\[ f_1 = x^2 + y^2, \ f_2 = xy \]
\[ I = \langle f_1, f_2 \rangle, \text{ lexicographic order } (x > y) \]

Then, \( \{ f_1, f_2 \} \) is not a Gröbner basis of \( I \) since

- \( f = yf_1 - xf_2 = y(x^2 + y^2) - x \cdot xy = y^3 \) belongs to \( I \).
- So, \( \text{in}_<(f) = y^3 \) belongs to \( \text{in}_<(I) \).
- \( \text{in}_<(f) = y^3 \) is divided by neither \( \text{in}_<(f_1) = x^2 \) nor \( \text{in}_<(f_2) = xy \).

\[ \rightarrow \text{ In this case, } \{ f_1, f_2, f \} \text{ is a Gröbner basis of } I. \]
Basic properties of a Gröbner basis $\mathcal{G}$ of an ideal $I$:

- Always exists.
- Not unique.
- $\mathcal{G}$ generates $I$.
- For any nonzero polynomial $f \in K[x_1, \ldots, x_n]$, the remainder of $f$ with respect to $\mathcal{G}$ is unique.
- For any nonzero polynomial $f \in K[x_1, \ldots, x_n]$, $f \in I \iff$ the remainder of $f$ with respect to $\mathcal{G}$ is 0.
Special Gröbner bases

\[ \mathcal{G} = \{g_1, \ldots, g_t\} : \text{a Gröbner basis of an ideal } I \]

**Definition**

\( \mathcal{G} \) is called **minimal** if each \( g_i \) is monic and

- \( \text{in}_< (g_j) \) is not divided by \( \text{in}_< (g_i) \) if \( i \neq j \).

\( \mathcal{G} \) is minimal \( \iff \mathcal{G} \setminus \{g_i\} \) is not a Gröbner basis for \( \forall i \).

**Definition**

\( \mathcal{G} \) is called **reduced** if each \( g_i \) is monic and

- Any monomial in \( g_j \) is not divided by \( \text{in}_< (g_i) \) if \( i \neq j \).

If we fix an ideal and a monomial order, then the reduced Gröbner basis exists (and **unique**).
Example

\[ l = \langle x_1 - x_2, x_1 - x_3 \rangle \]
\[ \langle \ \rangle \text{: lexicographic order} \]
\[ \text{in}_{<_{\text{lex}}} (l) = \langle x_1, x_2 \rangle \]
\[ \{ x_1 - x_2, x_1 - x_3, x_2 - x_3 \} : \text{Gröbner basis , not minimal} \]
\[ \{ x_1 - x_2, x_2 - x_3 \} : \text{minimal Gröbner basis , not reduced} \]
\[ \{ x_1 - x_3, x_2 - x_3 \} : \text{reduced Gröbner basis} \]
(0 ≠) \( f, g \in K[x_1, \ldots, x_n] \)
\( m := \text{LCM}(\text{in}_<(f), \text{in}_<(g)) \)
\( f = c_f \cdot \text{in}_<(f) + \cdots \)
\( g = c_g \cdot \text{in}_<(g) + \cdots \)

Then, we define the **S-polynomial** of \( f \) and \( g \) by

\[
S(f, g) := \frac{m}{c_f \cdot \text{in}_<(f)} f - \frac{m}{c_g \cdot \text{in}_<(g)} g.
\]

**Example**

\( f = x_1 x_4 - x_2 x_3, \ g = 2 x_4 x_7 - x_5 x_6, \ \text{<}_{\text{lex}}: \text{lexicographic order} \)

\[
S(f, g) = \frac{x_1 x_4 x_7}{x_1 x_4} (x_1 x_4 - x_2 x_3) - \frac{x_1 x_4 x_7}{2 x_4 x_7} (2 x_4 x_7 - x_5 x_6)
\]

\[
= -x_2 x_3 x_7 + \frac{1}{2} x_1 x_5 x_6
\]
Buchberger criterion

Theorem

\[ l = \langle g_1, \ldots, g_t \rangle \subset K[x_1, \ldots, x_n] \]

Then, \( \{ g_1, \ldots, g_t \} \) is a Gröbner basis of \( l \)

\[ \iff \]

The remainder of \( S(g_i, g_j) \) with respect to \( \{ g_1, \ldots, g_t \} \) is 0 for all \( i \neq j \).
Buchberger algorithm

**Input:** \( g_1, \ldots, g_t \in K[x_1, \ldots, x_n] \), monomial order \(<\)

**Output:** A Gröbner basis \( G \) of \( I = \langle g_1, \ldots, g_t \rangle \subset K[x_1, \ldots, x_n] \) w.r.t. \(<\)

**Step 1.** \( G = \{g_1, \ldots, g_t\} \)

**Step 2.** Apply Buchberger criterion to \( G \).

**Step 3.** If it satisfies the condition of the criterion, then \( G \) is a Gröbner basis .
If not, then there exists a nonzero remainder. Add it to \( G \) and back to step 2.
Example

\[ f = x_1 x_4 - x_2 x_3, \quad g = x_4 x_7 - x_5 x_6 \]
\[ I = \langle f, g \rangle \]

lexicographic order \((x_1 > \cdots > x_n)\)

\[ G = \{ f, g \} \]

\[ S(f, g) = x_7 f - x_1 g = x_1 x_5 x_6 - x_2 x_3 x_7 \]

A remainder of \(S(f, g)\) w.r.t. \(\{f, g\}\) is \(x_1 x_5 x_6 - x_2 x_3 x_7 =: h\)

\[ G = \{ f, g, h \} \]

\[ S(f, h) = x_5 x_6 f - x_4 h = x_2 x_3 x_4 x_7 - x_2 x_3 x_5 x_6 = x_2 x_3 g \]

\[ S(g, h) \rightarrow \text{remainder w.r.t. } G \text{ is zero.} \]

Thus, \(\{f, g, h\}\) is a Gröbner basis of \(I\).
Improving the efficiency of Buchberger algorithm

Proposition

\((0 \neq) f, g \in K[x_1, \ldots, x_n]\)
\[
\text{GCD}(\text{in}_{<}(f), \text{in}_{<}(g)) = 1
\]
\[\implies \text{the remainder of } S(f, g) \text{ with respect to } \{f, g\} \text{ is } 0.\]

- Strategies for selecting S-polynomials
  - Sugar Selection Strategy (in Proc. ISSAC 1991)
    \[\implies \text{first implemented in CoCoA}\]
- Homogenization
  (to avoid unnecessary intermediate coefficient swells)
  \[\ldots\]
Elimination theorem

**Theorem**

0 < m < n: integers

<: monomial order on $K[x_1, \ldots, x_n]$

\(\{0\} \neq I \subset K[x_1, \ldots, x_n]: ideal\)

$G$: Gröbner basis of $I$ w.r.t. $<$

If $<$ satisfies the condition

$$g \in G, \text{in}_<(g) \in K[x_1, \ldots, x_m] \implies g \in K[x_1, \ldots, x_m],$$

then $G \cap K[x_1, \ldots, x_m]$ is a Gröbner basis of $I \cap K[x_1, \ldots, x_m]$. 

Example

In order to solve the system of equations

\[
\begin{align*}
    f_1 &= x^2 + y + z - 1 = 0, \\
    f_2 &= x + y^2 + z - 1 = 0, \\
    f_3 &= x + y + z^2 - 1 = 0,
\end{align*}
\]

we compute a Gröbner basis of the ideal \( \langle f_1, f_2, f_3 \rangle \) with respect to the lexicographic order \( \langle \text{lex} \ (x > y > z) \rangle \):

\[
\begin{align*}
    g_1 &= x + y + z^2 - 1, \\
    g_2 &= y^2 - y - z^2 + z, \\
    g_3 &= 2yz^2 + z^4 - z^2, \\
    g_4 &= z^6 - 4z^4 + 4z^3 - z^2.
\end{align*}
\]

\( \langle \text{lex} \rangle \) satisfies the condition in Elimination Theorem. Since \( \langle f_1, f_2, f_3 \rangle = \langle g_1, g_2, g_3, g_4 \rangle \) holds, \( f_1 = f_2 = f_3 = 0 \) and \( g_1 = g_2 = g_3 = g_4 = 0 \) have the same solutions.
**Toric ideals**

\( \mathbb{Z}^{d \times n} \): the set of all \( d \times n \) integer matrices
\[ A = (a_1, \ldots, a_n) \in \mathbb{Z}^{d \times n} \]

- \( A \) is called a **configuration** if \( \exists \mathbf{w} \in \mathbb{R}^d \) s.t. \( \mathbf{w} \cdot a_1 = \cdots = \mathbf{w} \cdot a_n = 1 \).
- We usually assume that \( A \) is a configuration.

\( K \): field (e.g., \( K = \mathbb{C} \))

\( K[X] = K[x_1, \ldots, x_n] \): poly. ring in \( n \) variables over \( K \)

- \( \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0} \implies X^\mathbf{u} = x_1^{u_1} \cdots x_n^{u_n} \)

\[ I_A = \langle X^\mathbf{u} - X^\mathbf{v} \in K[X] \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0}, A\mathbf{u} = A\mathbf{v} \rangle \]

\[ = \langle X^{\mathbf{u}^+} - X^{\mathbf{u}^-} \in K[X] \mid \mathbf{u} \in \mathbb{Z}^n, A\mathbf{u} = \mathbf{0} \rangle \]

Toric ideal of \( A \)
Example

\[ A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix} \]

\[ I_A = \langle x_1x_5 - x_2x_4, \ x_1x_6 - x_3x_4, \ x_2x_6 - x_3x_5 \rangle \]

(For example, \( x_1x_5 - x_2x_4 \in I_A \) since \( A \begin{pmatrix}
1 \\
-1 \\
0 \\
-1 \\
1 \\
0
\end{pmatrix} = 0 \).)

\[ K[A] = K[t_1t_3, \ t_1t_4, \ t_1t_5, \ t_2t_3, \ t_2t_4, \ t_2t_5] \cong K[X]/I_A \]
Basic properties

Properties of toric ideals:

- prime ideal
- The reduced Gröbner basis of $I_A$ consists of binomials.
- $A$ is a configuration
  $\iff I_A$ is homogeneous w.r.t. a usual grading
- $A \in \mathbb{Z}_{\geq 0}^n$
  $\implies I_A$ is homogeneous w.r.t. some positive grading
- If each $a_i$ is a nonnegative integer vector, then
  $$I_A = \langle x_1 - T^{a_1}, \ldots, x_n - T^{a_n} \rangle \cap K[X].$$

So, we can compute a Gröbner basis of $I_A$ by Elimination theorem.
However, this method is not effective in practice.
Algorithm computing generators of $I_A$

**Lemma**

$J \subset K[X]:$ homogeneous ideal

$\prec: reverse$ lexicographic order

$\mathcal{G}:$ the reduced Gröbner basis of $J$ w.r.t. $\prec$

$(J : x_n^{\infty}) := \{ f \in K[X] \mid \exists r \in \mathbb{N} \text{ s.t. } x_n^r f \in J \}$

Then, a GB of $(J : x_n^{\infty})$ w.r.t. $\prec$ is obtained by dividing each $g \in \mathcal{G}$ by the highest power of $x_n$ that divides $g$.

**Proposition**

$A \in \mathbb{Z}^{d \times n}:$ configuration

$B: lattice basis of \{ u \in \mathbb{Z}^n \mid Au = 0 \}$

$J := \left< X^u_+ - X^u_- \mid u \in B \right>$

Then $I_A = (J : (x_1 \cdots x_n)^{\infty}) = ((\cdots(J : x_1^{\infty}) : x_2^{\infty}) \cdots) : x_n^{\infty})$
Three breakthroughs

Toric ideals

0. Commutative algebra

- Toric ideals have been studied by commutative algebraists for a long time.

- For example,

  J. Herzog
  Generators and relations of abelian semigroups and semigroup rings

  is an early reference in commutative algebra.
Three breakthroughs

1. Integer programming

P. Conti and C. Traverso
Buchberger algorithm and integer programming
in Proceedings of AAECC-9 (New Orleans)
Springer LNCS 539 (1991), 130 – 139.
Three breakthroughs

2. Triangulations of convex polytopes.

- I. M. Gel’fand, A. V. Zelevinskii and M. M. Kapranov
  Hypergeometric functions and toral manifolds

- B. Sturmfels
  Gröbner bases of toric varieties
Three breakthroughs

3. Conditional test of contingency tables
   (Markov chain Monte Carlo method)

P. Diaconis and B. Sturmfels
Algebraic algorithms for sampling from conditional distributions
(Received June 1993; revised April 1997.)
Three breakthroughs

One can study three breakthroughs in

B. Sturmfels
Gröbner bases and convex polytopes

See also

D. Cox, J. Little and D. O’Shea
Using algebraic geometry

T. Hibi (Ed.)
Gröbner bases –Statistics and Software Systems–
### B.1. Integer programming

**Example (CLO, “Using algebraic geometry”)**

Each pallet from a customer A : 400 kg, 2 m³  
Each pallet from a customer B : 500 kg, 3 m³  
The customer A will pay $ 11 for each pallet, and  
the customer B will pay $ 15 for each pallet.  
We use trucks that can carry any load  
up to 3700 kg, and up to 20 m³.

How to maximize the revenues generated?

Subject to \[
\begin{align*}
4a + 5b & \leq 37 \\
2a + 3b & \leq 20 , \\
a, b & \geq 0
\end{align*}
\]

find integers \(a, b\) which maximize \(11a + 15b\).
Example

\[
\begin{aligned}
\text{Subject to } & \\
& \begin{cases}
4a + 5b \leq 37 \\
2a + 3b \leq 20 , \\
a, b \geq 0
\end{cases}
\end{aligned}
\]

find integers \( a, b \) which maximize \( 11a + 15b \).
Subject to \[
\begin{align*}
4a + 5b &\leq 37 \\
2a + 3b &\leq 20 , \\
a, b &\geq 0
\end{align*}
\]
find integers $a, b$ which maximize $11a + 15b$.

↓ the standard form

Subject to \[
\begin{align*}
4a + 5b + c & = 37 \\
2a + 3b + d & = 20 , \\
a, b, c, d &\geq 0
\end{align*}
\]
find integers $a, b, c$, and $d$ which minimize $-11a - 15b$. 
Conti–Traverso algorithm

Subject to $A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 37 \\ 20 \end{pmatrix}$ where $A = \begin{pmatrix} 4 & 5 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$.

$w := (-11, -15, 0, 0) + 2 \cdot (6, 8, 1, 1) = (1, 1, 2, 2)$.

$G = \{ x_3^4 x_4^2 - x_1, x_2 x_3^3 x_4 - x_1^2, x_1 x_3 x_4 - x_2, x_1^4 x_4 - x_2^3 x_3, x_2^2 x_3^2 - x_1^3 \}$

is a Gröbner basis of $I_A$ with respect to $<_w$.

$(a, b, c, d) = (0, 0, 37, 20)$ satisfies the constraints. Therefore, we compute the remainder of $x_3^{37} x_4^{20}$ w.r.t. $G$.

The remainder is $x_1^4 x_2^4 x_3$.

Hence, $(a, b, c, d) = (4, 4, 1, 0)$ is a solution.

Answer: Four pallets from A and four pallets from B.
B.2. Triangulations of convex polytopes

In this section, we always assume that
\( A = (a_1, \ldots, a_n) \in \mathbb{Z}^{d \times n} \) is a configuration, and often identify \( A \) with the set \( \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d \).

**Definition**

\[
\text{Conv}(A) := \left\{ \sum_{i=1}^{n} r_i a_i \in \mathbb{Q}^d \mid 0 \leq r_i \in \mathbb{Q}, \sum_{i=1}^{n} r_i = 1 \right\}
\]

the convex hull of \( A \).
Example

\[
A = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}.
\]

Then, \(\text{Conv}(A)\) is
**Definition**

An integral convex polytope $P$ is called **simplex** if the number of vertex of $P$ is $1 + \dim P$.
(ex. line, triangle, tetrahedron.)

**Definition**

A covering $\Delta$ of $A$ is a set of simplices whose vertices belong to $A$ such that $\text{Conv}(A) = \bigcup_{F \in \Delta} F$.

**Definition**

A covering $\Delta$ of $A$ is called a **triangulation** if

1. $F'$ is a face of $F \in \Delta \implies F' \in \Delta$
2. $F, F' \in \Delta \implies F \cap F'$ is a face of $F$ and a face of $F'$. 
### Initial complex

**Definition (initial complex)**

\[ A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d \]

\( < \) : monomial order

\[
\Delta(in_<(I_A)) := \left\{ \text{Conv}(B) \ \bigg| \  \prod_{a_i \in B} x_i \notin \sqrt{in_<(I_A)} \right\}
\]

**Theorem**

\( \Delta(in_<(I_A)) \) is a triangulation of \( A \).

- (Gel’fand et al.) For \( w \in \mathbb{R}^n \), a triangulation \( \Delta_w \) is defined geometrically. (regular triangulation)

- (Sturmfels) We have \( \Delta(in_w(I_A)) = \Delta_w \).
Example

\[ A = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} \]

\[ I_A = \langle x_1x_2 - x_3x_5, x_1x_4 - x_2x_3, x_2^2 - x_4x_5 \rangle \]

\(<_1: \text{lexicographic order}(x_2 > x_1 > x_3 > x_4 > x_5)\)

Gröbner bases of \( I_A \) with respect to \(<_1\) is

\[ \{ x_1x_2 - x_3x_5, x_2x_3 - x_1x_4, x_2^2 - x_4x_5, x_1^2x_4 - x_3^2x_5 \} \]

\[ \text{in}_{<_1}(I_A) = \langle x_1x_2, x_2x_3, x_2^2, x_1^2x_4 \rangle, \quad \sqrt{\text{in}_{<_1}(I_A)} = \langle x_1x_4, x_2 \rangle \]
Examples

Example

\[ A = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}, \quad \sqrt{\text{in}_{<1}(I_A)} = \langle x_1 x_4, x_2 \rangle \]
Examples

Example

\[ A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]

\[ I_A = \langle x_1 x_2 - x_3 x_5, x_1 x_4 - x_2 x_3, x_2^2 - x_4 x_5 \rangle \]

\(<_2: \text{lexicographic order}(x_5 > x_3 > x_4 > x_2 > x_1)\)

Gröbner bases of \( I_A \) with respect to \(<_2\) is

\[ \{ x_3 x_5 - x_1 x_2, x_2 x_3 - x_1 x_4, x_4 x_5 - x_2^2 \} \]

\[ in_{<_2}(I_A) = \sqrt{in_{<_2}(I_A)} = \langle x_2 x_3, x_3 x_5, x_4 x_5 \rangle \]
Example

\[
A = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}, \quad \sqrt{\text{in}_{\leq 2}(I_A)} = \langle x_2x_3, x_3x_5, x_4x_5 \rangle
\]
Definition

A covering (triangulation) $\Delta$ of $A$ is called **unimodular**, if for the vertex set $B$ of any maximal simplex in $\Delta$, we have $\mathbb{Z}A = \mathbb{Z}B$,

$\mathbb{Z}A = \left\{ \sum_{i=1}^{n} z_i a_i \mid z_i \in \mathbb{Z} \right\}$.

Theorem

$\Delta(\text{in}_{<}(I_A))$ is unimodular $\iff \sqrt{\text{in}_{<}(I_A)} = \text{in}_{<}(I_A)$
Important properties

(i) $A$ is unimodular (any triangulation of $A$ is unimodular)
   $(\iff \sqrt{\text{in}_<(I_A)} = \text{in}_<(I_A)$ for any $<$)

(ii) $A$ is compressed
     $(\iff \sqrt{\text{in}_<(I_A)} = \text{in}_<(I_A)$ for any reverse lex. order $<$)

(iii) $A$ has a regular unimodular triangulation
     $(\iff \sqrt{\text{in}_<(I_A)} = \text{in}_<(I_A)$ for some $<$)

(iv) $A$ has a unimodular triangulation

(v) $A$ has a unimodular covering

(vi) $K[A]$ is normal $(\iff \mathbb{Z}_{\geq 0}A = \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A)$

Then, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$ hold.
However, the converse of them are false in general.
Edge polytopes

$G$: finite connected graph on the vertex set $\{1, 2, \ldots, d\}$

$E(G) = \{e_1, \ldots, e_n\}$: the edge set of $G$

(no loop, no multiple edges)

For each edge $e = \{i, j\} \in E(G)$, let $\rho(e) := e_i + e_j \in \mathbb{Z}^d$.

$A_G := (\rho(e_1), \ldots, \rho(e_n)) \in \mathbb{Z}^{d \times n}$

$\text{Conv}(A_G)$ is called an edge polytope of $G$. 
Edge polytopes

Theorem (O–Hibi (1998), Simis et al. (1998))

For a finite connected graph $G$, TFAE:

1. $K[A_G]$ is normal;
2. $A_G$ has a unimodular covering;
3. For any two odd cycles $C$ and $C'$ of $G$ without common vertices, there exists an edge of $G$ which joins a vertex of $C$ with a vertex of $C'$. 
An interesting edge polytope

Example (O–Hibi (1999))

Let $G$ be the following graph. Then,

- For any monomial order $<, \sqrt{\text{in}<(I_{AG})} \neq \text{in}<(I_{AG})$
- $A_G$ has a unimodular triangulation.

(Checked by the software PUNTOS by De Loera.)
B.3. Contingency tables

$5 \times 5$ contingency table:

<table>
<thead>
<tr>
<th>algebra \ statistics</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>total</td>
<td>10</td>
<td>6</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>26</td>
</tr>
</tbody>
</table>

Is there a correlation between the two scores?
Markov chain Monte Carlo method

\[ F = \begin{cases} T = (t_{ij}) & t_{ij} \\
\end{cases} \]

\[
\begin{array}{c|ccccc}
& 10 & 6 & 5 & 2 & 3 \\
\hline
1 & 4 & 14 & 5 & 2 & 1 \\
2 & & & & & \\
\end{array}
\]

, \quad 0 \leq t_{ij} \in \mathbb{Z}

By a random walk on \( F \), we sample elements of \( F \) and compare certain features (\( \chi^2 \)-statistics).
(In this example, \( \#|F| = 229, 174 \).)
Markov chain Monte Carlo method

For example, fix $\alpha_i, \beta_j$ such that $\sum_i \alpha_i = \sum_j \beta_j$ and let

$$F = \left\{ T = (t_{ij}) \left| \begin{array}{ccc}
    t_{11} & t_{12} & t_{13} \\
    t_{21} & t_{22} & t_{23} \\
    \beta_1 & \beta_2 & \beta_3
\end{array} \right| \begin{array}{c}
    \alpha_1 \\
    \alpha_2
\end{array}, \quad 0 \leq t_{ij} \in \mathbb{Z} \right\}.$$  

Then by adding or subtracting one of the elements of

$$M = \left\{ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right\},$$

any of two elements $T, T'$ of $F$ are connected:

$$T = T_0 \longrightarrow T_1 \in F \longrightarrow T_2 \in F \longrightarrow \cdots \longrightarrow T_s = T'.$$

Such an $M$ is called a Markov basis.
Markov bases and toric ideals

Example (continued)

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad A \begin{pmatrix} t_{11} \\ t_{12} \\ t_{13} \\ t_{21} \\ t_{22} \\ t_{23} \end{pmatrix} = \begin{pmatrix} t_{11} + t_{12} + t_{13} \\ t_{21} + t_{22} + t_{23} \\ t_{11} + t_{21} \\ t_{12} + t_{22} \\ t_{13} + t_{23} \end{pmatrix} \]

\[ I_A = \langle x_1 x_5 - x_2 x_4, \; x_1 x_6 - x_3 x_4, \; x_2 x_6 - x_3 x_5 \rangle \]

\[ \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \} \]
<table>
<thead>
<tr>
<th>Theorem (Diaconis–Sturmfels)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $M$ be a finite set of integer matrices. Then, $M$ is a Markov basis if and only if $I_A$ is generated by the corresponding binomials.</td>
</tr>
</tbody>
</table>

- **2 way contingency tables:**
  It is known that $I_A$ has a quadratic Gröbner basis.

- **≥ 3 way contingency tables:**
  Except for some classes, the set of generators of $I_A$ is unknown and it is not easy to compute in general. (You should try to use powerful software *4ti2.*).
No $n$ way interaction models

For $r_1 \times r_2 \times \cdots \times r_\ell$ contingency table($r_1 \geq r_2 \geq \cdots \geq r_\ell \geq 2$)

$$T = (t_{i_1 i_2 \cdots i_\ell})_{i_k=1,2,\ldots,r_k}, \quad 0 \leq t_{i_1 i_2 \cdots i_\ell} \in \mathbb{Z},$$

we associates a configuration $A_{r_1 r_2 \cdots r_\ell}$ consisting of the following vectors:

$$e^{(1)}_{i_2 i_3 \cdots i_\ell} \oplus e^{(2)}_{i_1 i_3 \cdots i_\ell} \oplus \cdots \oplus e^{(\ell)}_{i_1 i_2 \cdots i_{\ell-1}}$$

where each $i_k$ belongs to $\{1, 2, \ldots, r_k\}$ and $e^{(k)}_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_\ell}$ is a unit vector in $\mathbb{R}^{r_1 \cdots r_{k-1} r_{k+1} \cdots r_\ell}$. 
A_{222}
## Classification

<table>
<thead>
<tr>
<th>$r_1 \times r_2$</th>
<th>unimodular</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 \times r_2 \times 2 \times \cdots \times 2$</td>
<td></td>
</tr>
<tr>
<td>$r_1 \times 3 \times 3$</td>
<td>compressed, not unimodular</td>
</tr>
<tr>
<td>$5 \times 5 \times 3$</td>
<td>normal (4ti2 &amp; Normaliz)</td>
</tr>
<tr>
<td>$5 \times 4 \times 3$</td>
<td>not compressed</td>
</tr>
<tr>
<td>$4 \times 4 \times 3$</td>
<td></td>
</tr>
<tr>
<td>otherwise, i.e.,</td>
<td>not normal</td>
</tr>
<tr>
<td>$\ell \geq 4$ and $r_3 \geq 3$</td>
<td></td>
</tr>
<tr>
<td>$\ell = 3$ and $r_3 \geq 4$</td>
<td></td>
</tr>
<tr>
<td>$\ell = 3$, $r_3 = 3$, $r_1 \geq 6$ and $r_2 \geq 4$</td>
<td></td>
</tr>
</tbody>
</table>
Decompositions/constructions

- Algebras of Veronese type
- Segre–Veronese configurations (O–Hibi 2000)
- Higher Lawrence configurations (Santos–Sturmfels 2003) \( \rightarrow \) \( N \)-fold configurations
- Toric fiber product (Sullivant 2007)
Segre–Veronese configurations

\( \tau \geq 2, n: \) integers
\( \mathbf{b} = (b_1, \ldots, b_n), \mathbf{c} = (c_1, \ldots, c_n), \mathbf{p} = (p_1, \ldots, p_n), \mathbf{q} = (q_1, \ldots, q_n) : \) integer vectors satisfying

1. \( 0 \leq c_i \leq b_i \) for all \( 1 \leq i \leq n \)
2. \( 1 \leq p_i \leq q_i \leq d \) for all \( 1 \leq i \leq n \)

Let \( A \) be the matrix whose columns are all vectors
\( (f_1, \ldots, f_d) \in \mathbb{Z}_{\geq 0}^d \) s.t.

1. \( \sum_{j=1}^{d} f_j = \tau \)
2. \( c_i \leq \sum_{j=p_i}^{q_i} f_j \leq b_i \) for all \( 1 \leq i \leq n \)

\( A \) is called Segre–Veronese configuration.
Segre–Veronese configurations

Example

\( \tau = 2, \ n = 2, \ d = 5 \)

\( b = (1, 1), \ c = (0, 0), \ p = (1, 3), \ q = (2, 5) \)

Let \( A \) be the matrix whose columns are all vectors

\( (f_1, \ldots, f_5) \in \mathbb{Z}_{\geq 0}^5 \) s.t.

1. \( f_1 + f_2 + f_3 + f_4 + f_5 = 2 \)
2. \( 0 \leq f_1 + f_2 \leq 1 \)
3. \( 0 \leq f_3 + f_4 + f_5 \leq 1 \)

Then

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}.
\]
Segre–Veronese configurations

Theorem

Suppose that $A$ is a Segre–Veronese configuration. Then the toric ideal $I_A$ has a quadratic Gröbner basis.

S. Aoki, T. Hibi, H. Ohsugi and A. Takemura
Markov basis and Gröbner basis of Segre-Veronese configuration for testing independence in group-wise selections

S. Aoki, T. Otsu, A. Takemura and Y. Numata
Statistical Analysis of Subject Selection Data in NCUEE Examination
Oyo Tokeigaku, 39, (2)–(3), 2010, 71–100. (Japanese)
Nested configurations

\[ A = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{d \times n} : \text{configuration} \]

For each \( i = 1, 2, \ldots, d \):
\[ B_i \in \mathbb{Z}_{\geq 0}^{\mu_i \times \lambda_i} : \text{configuration} \]

\[ K \left[ u_1^{(i)}, \ldots, u_{\mu_i}^{(i)} \right] : \text{polynomial ring in } \mu_i \text{ variables} \]

\[ K[B_i] = K \left[ m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)} \right] \subset K \left[ u_1^{(i)}, \ldots, u_{\mu_i}^{(i)} \right] \]

\[ K[A(B_1, \ldots, B_d)] := K \left[ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \middle| \begin{array}{c} r \in \mathbb{N} \\ \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_r} \in A \\ 1 \leq j_k \leq \lambda_{i_k} \text{ for } \forall k \end{array} \right] \]

The configuration \( A(B_1, \ldots, B_d) \) is called the \textbf{nested configuration} of \( A, B_1, \ldots, B_d \).
Example

\[ A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ A(B_1, B_2) = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

- You have 2 coupons.
- Shop \( \alpha \) accepts at most 2 coupons.
- Shop \( \beta \) accepts at most 1 coupon.
- Each shop has 2 different items.
- A coupon allows you to buy 2 items at a discount at \( \alpha \).
- A coupon allows you to buy 1 item at a discount at \( \beta \).
Nested configurations

**Theorem**

Let $n \geq 2$.

- $I_A, I_{B_1}, \ldots, I_{B_d}$ have Gröbner bases of degree $\leq n$  
  $\implies I_{A(B_1,\ldots,B_d)}$ has a Gröbner basis of degree $\leq n$.

- $I_A, I_{B_1}, \ldots, I_{B_d}$ have squarefree initial ideals  
  $\implies I_{A(B_1,\ldots,B_d)}$ has a squarefree initial ideal.

**Theorem**

$K[A], K[B_1], \ldots, K[B_d]$ are normal  
$\implies K[A(B_1,\ldots,B_d)]$ is normal.

(The converse is not true in general.)
References

- S. Aoki, T. Hibi, H. Ohsugi and A. Takemura
  Gröbner bases of nested configurations

- H. Ohsugi and T. Hibi
  Toric rings and ideals of nested configurations

- T. Shibuta
  Gröbner bases of contraction ideals
0. Quadratic Gröbner bases

The following properties of Gröbner bases of toric ideals are studied by many researchers:

(i) There exists a monomial order such that a Gröbner basis of $I_A$ consists of quadratic binomials.
(ii) $K[A]$ is "Koszul algebra."
(iii) $I_A$ is generated by quadratic binomials.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold.
But neither (ii) $\Rightarrow$ (i) nor (iii) $\Rightarrow$ (ii) holds in general.
**Example (O–Hibi (1999))**

Let $G$ be the following graph. Then,

- $I_{A_G}$ is generated by quadratic binomials.
- $K[A_G]$ is not Koszul.
- Hence, for any monomial order $<$, the reduced Gröbner basis of $I_{A_G}$ is not quadratic.
Infinite family of counterexamples


Using software \texttt{Risa/Asir, Macauley2, CaTS,...}, we checked that there are a lot of graphs of \( \leq 8 \) vertices whose edge polytope is a counterexample. Moreover, we proved that

\textbf{Theorem}

\( n \geq 5 \)

\( C_n : \text{cycle of length } n \)

\( K_{n+1} : \text{the complete graph with } n + 1 \text{ vertices} \)

\( G := K_{n+1} - E(C_n) \)

Then,

\( I_{AG} \) is generated by quadratic binomials

\( I_{AG} \) has no quadratic Gröbner basis.
Configurations arising from root systems

Gel’fand–Graev–Postnikov (1997)

\[ A_{d-1} = \{ e_i - e_j \mid 1 \leq i < j \leq d \} \quad (e_i \in \mathbb{Z}^d \text{ is a unit vector}) \]

\[ \widetilde{A}_{d-1} = \begin{pmatrix} 0 & \vdots & A_{d-1} \\ 0 & 1 & \cdots & 1 \end{pmatrix} \]

There exists a monomial order such that a Gröbner basis of \( I_{\widetilde{A}_{d-1}} \) consists of quadratic binomials. (They constructed a “regular unimodular triangulation.”)

O–Hibi (2002)

\[ D_d = \{ e_i + e_j \mid 1 \leq i < j \leq d \} \cup A_{d-1} \]
\[ B_d = \{ e_1, \ldots, e_d \} \cup D_d \]
\[ C_d = \{ 2e_1, \ldots, 2e_d \} \cup D_d \]
Toric ideals arising from matroids

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ where

- Each $B_i$ is an $r$-subset of $\{1, 2, \ldots, d\}$;
- (Basis Exchange Axiom)
  
  For each $1 \leq i, j \leq n$, for $\forall x \in B_i \setminus B_j$, 
  
  $\exists y \in B_j \setminus B_i$ s. t. $(B_i \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Let $A_\mathcal{B} = (a_1, \ldots, a_n) \in \mathbb{Z}^{d \times n}$ where $a_i = \sum_{j \in B_i} e_j \in \mathbb{R}^d$.

**Conjecture (White (1980))**

$I_{A_\mathcal{B}}$ is generated by quadratic binomials.

**Conjecture**

$I_{A_\mathcal{B}}$ has a quadratic Gröbner basis.